

# ASYMPTOTIC EXPANSION OF THE WAVELET TRANSFORM WITH ERROR TERM

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## Abstract

Using Wong's technique asymptotic expansion for the wavelet transform is derived and thereby asymptotic expansions for Morlet wavelet transform, Mexican Hat wavelet transform and Haar wavelet transform are obtained.

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## 1 Introduction

The wavelet transform of  $f$  with respect to the wavelet  $\psi$  is defined by

$$(W_\psi f)(b, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt, \quad b \in \mathbb{R}, a > 0, \quad (1)$$

provided the integral exists [1]. Using Fourier transform it can also be expressed as

$$(W_\psi f)(b, a) = \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{ib\omega} \overline{\hat{\psi}(a\omega)} d\omega, \quad (2)$$

where

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-it\omega} f(t) dt.$$

Asymptotic expansion with explicit error term for the general integral

$$I(x) = \int_0^{\infty} f(t) h(xt) dx, \quad (3)$$

where  $h(t)$  is an oscillatory function, was obtained by Wong [3], [4] under different conditions on  $g$  and  $h$ . Then the asymptotic expansion for (2) can be obtained by setting  $g(t) = e^{ibt} \hat{f}(t)$  for fixed  $b \in \mathbb{R}$ . Let us recall basic results from [4] which will be used in the present investigation. Here we assume that  $g(t)$  has an expansion of the form

$$\begin{aligned} g(t) &\sim \sum_{s=0}^{\infty} c_s t^{s+\lambda-1} \quad \text{as } t \rightarrow 0, \\ &= \sum_{s=0}^{n-1} c_s t^{s+\lambda-1} + g_n(t) \end{aligned} \quad (4)$$

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<sup>1</sup> This work is contained in the research monograph "The Wavelet Transform" by Prof. R S Pathak and edited by Prof. C. K. Chui (Stanford University, U.S.A.) and published by Atlantis Press/World Scientific (2009), ISBN: 978-90-78677-26-0, pp:154-164

where  $0 < \lambda \leq 1$ . Regarding the function  $h$ , we assume that as  $t \rightarrow 0+$ ,

$$h(t) = O(t^\rho), \quad \rho + \lambda > 0, \quad (5)$$

and that as  $t \rightarrow +\infty$ ,

$$h(t) \sim \exp(i\tau t^p) \sum_{s=0}^{\infty} b_s t^{-s-\beta}, \quad (6)$$

where  $\tau \neq 0$  is real,  $p \geq 1$  and  $0 < \beta \leq 1$ . Let  $M[h; z]$  denote the generalized Mellin transform of  $h$  defined by

$$M[h; z] = \lim_{\varepsilon \rightarrow 0+} \int_0^\infty t^{z-1} h(t) \exp(-\varepsilon t^p) dt. \quad (7)$$

This, together with (48) and [4, p.216], gives

$$I(x) = \sum_{s=0}^{n-1} c_s M[h; s + \lambda] x^{-s-\lambda} + \delta_n(x), \quad (8)$$

where

$$\delta_n(x) = \lim_{\varepsilon \rightarrow 0+} \int_0^\infty g_n(t) h(xt) \exp(-\varepsilon t^p) dt. \quad (9)$$

If we now define recursively  $h^\circ(t) = h(t)$  and

$$h^{(-j)}(t) = - \int_t^\infty h^{(-j+1)}(u) du, \quad j = 1, 2, \dots,$$

Repeated integration by part, we have

$$h^{(-j)}(t) \sim \exp(i\tau t^p) \sum_{s=0}^{\infty} b_s^{(j)} t^{-\mu_{s,j}}, \quad \text{as } t \rightarrow \infty, \quad (10)$$

where  $b_s^{(j)}$  are some constants and for each  $j$ , and  $\mu_{s,j}$  is a monotonically increasing sequence of positive numbers depending on  $p$  and  $\beta$ .

Then conditions of validity of aforesaid results are given by the following [4, Theorem 6, p.217]:

**Theorem 1.** . Assume that (i)  $g^{(m)}(t)$  is continuous on  $(0, \infty)$ , where  $m$  is a non-negative integer; (ii)  $g(t)$  has an expansion of the form (4), and the expansion is  $m$  times differentiable; (iii)  $h(t)$  satisfies (5) and (6) and (iv) and as  $t \rightarrow \infty$ ,  $t^{-\beta} g^{(j)}(t) = O(t^{-1-\varepsilon})$  for  $j = 0, 1, \dots, m$  and for some  $\varepsilon > 0$ . Under these conditions, the result (8) holds with

$$\delta_n(x) = \frac{(-1)^m}{x^m} \int_0^\infty g_n^{(m)}(t) h^{(-m)}(xt) dt, \quad (11)$$

where  $n$  is the smallest positive integer such that  $\lambda + n > m$ .

*Proof.* Integrating by part (9) we get

$$\begin{aligned} \int_0^\infty f_n(t)h(xt)e^{-\epsilon t^p} dt &= -\frac{1}{x} \int_0^\infty f_n(t)e^{-\epsilon t^p} h^{(-1)}(xt) dt \\ &\quad + \frac{\epsilon p}{x} \int_0^\infty f_n(t)h^{(-1)}(xt)t^{p-1}e^{-\epsilon t^p} dt \end{aligned} \quad (12)$$

the integrated term vanishing due to  $\rho+\lambda > 0$  and condition (iv) and the asymptotic behaviour in (10). The same reasoning, together with Lemma1 and Lemma2, ensures that the second term on the right-hand side of (12) tends to zero as  $\epsilon \rightarrow 0+$ .

Thus

$$\delta_n(x) = \left(-\frac{1}{x}\right) \lim_{\epsilon \rightarrow 0+} \int_0^\infty f'_n(t)h^{(-1)}(xt)e^{-\epsilon t^p} dt. \quad (13)$$

Repeated application of this technique shows that

$$\begin{aligned} \delta_n(x) &= \frac{(-1)^m}{x^m} \lim_{\epsilon \rightarrow 0+} \int_0^\infty f_n^{(m)}(t)h^{(-m)}(xt)e^{-\epsilon t^p} dt \\ &= \frac{(-1)^m}{x^m} \int_0^\infty f_n^{(m)}(t)h^{(-m)}(xt)e^{-\epsilon t^p} dt. \end{aligned} \quad (14)$$

The last equality again follows from Lemma1.  $\square$

The aim of the present paper is to derive asymptotic expansion of the wavelet transform given by (2) for large values of  $a$ , using formula (8). We also obtain asymptotic expansions for the special transforms corresponding to Morlet wavelet, Mexican hat wavelet and Haar wavelet.

## 2 Asymptotic expansion for large $a$

In this section using aforesaid technique, we obtain asymptotic expansion of  $(W_\psi f)(b, a)$  for large values of  $a$ , keeping  $b$  fixed. We have

$$\begin{aligned} (W_\psi f)(b, a) &= \frac{\sqrt{a}}{2\pi} \int_{-\infty}^\infty e^{ib\omega} \bar{\hat{\psi}}(a\omega) \hat{f}(\omega) d\omega \\ &= \frac{\sqrt{a}}{2\pi} \left\{ \int_0^\infty e^{ib\omega} \bar{\hat{\psi}}(a\omega) \hat{f}(\omega) d\omega \right. \\ &\quad \left. + \int_0^\infty e^{-ib\omega} \bar{\hat{\psi}}(-a\omega) \hat{f}(-\omega) d\omega \right\} \\ &= \frac{\sqrt{a}}{2\pi} (I_1 + I_2), \text{ say.} \end{aligned} \quad (15)$$

Let us set

$$h(\omega) = \bar{\hat{\psi}}(\omega). \quad (16)$$

Assume that

$$\overline{\hat{\psi}}(\omega) \sim \exp(i\tau\omega^p) \sum_{\tau=0}^{\infty} b_r \omega^{-r-\beta}, \quad \beta > 0, \omega \rightarrow +\infty, \tau \neq 0, p \geq 1, \quad (17)$$

and

$$\hat{f}(\omega) \sim \sum_{s=0}^{\infty} c_s \omega^{s+\lambda-1} \quad \text{as } \omega \rightarrow 0. \quad (18)$$

where  $0 < \lambda \leq 1$ . Also assume that as  $\omega \rightarrow 0$ ,

$$h(\omega) = \overline{\hat{\psi}}(\omega) = O(\omega^\rho), \quad \rho + \lambda > 0. \quad (19)$$

Then, as  $\omega \rightarrow 0$ ,

$$\begin{aligned} g(\omega) &:= e^{ib\omega} \hat{f}(\omega) \\ &\sim \sum_{s=0}^{\infty} c_s \omega^{s+\lambda-1} \sum_{r=0}^{\infty} \frac{(ib\omega)^r}{r!} \\ &= \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} c_s \frac{(ib)^r}{r!} \omega^{s+\lambda-1+r} \\ &= \sum_{s=0}^{\infty} \left\{ \sum_{r=0}^s \frac{(ib)^r}{r!} c_{s-r} \right\} \omega^{s+\lambda-1} \\ &= \sum_{s=0}^{\infty} d_s \omega^{s+\lambda-1}, \end{aligned} \quad (20)$$

where

$$d_s = \sum_{r=0}^s \frac{(ib)^r}{r!} c_{s-r}. \quad (21)$$

For each  $n \geq 1$ , we write

$$g(\omega) = \sum_{s=0}^{n-1} d_s \omega^{s+\lambda-1} + g_n(\omega). \quad (22)$$

The generalized Mellin transform of  $h$  is defined by

$$M[h; z_1] = \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \omega^{z_1-1} h(\omega) e^{-\varepsilon\omega} d\omega \quad (23)$$

Then by (8),

$$I_1(a) = \sum_{s=0}^{n-1} d_s M[h; s + \lambda] a^{-s-\lambda} + \delta_n^1(a), \quad (24)$$

where

$$\delta_n^1(a) = \lim_{\varepsilon \rightarrow 0+} \int_0^\infty g_n(\omega) h(a\omega) e^{-\varepsilon\omega} d\omega. \quad (25)$$

Also, from (19) we have

$$h(-\omega) = O(\omega^\rho), \quad \omega \rightarrow 0, \quad \rho + \lambda > 0 \quad (26)$$

and

$$M[h(-\omega); z_1] = \lim_{\varepsilon \rightarrow 0+} \int_0^\infty \omega^{z_1-1} h(-\omega) e^{-\varepsilon\omega} d\omega. \quad (27)$$

Hence

$$I_2(a) = \sum_{s=0}^{n-1} d_s (-1)^{s+\lambda+1} M[h(-\omega); s+\lambda] a^{-s-\lambda} + \delta_n^2(a), \quad (28)$$

where

$$\delta_n^2(a) = \lim_{\varepsilon \rightarrow 0+} \int_0^\infty g_n(-\omega) h(-a\omega) e^{-\varepsilon\omega} d\omega. \quad (29)$$

Finally, from (15), (24) and (28) we get the asymptotic expansion:

$$\begin{aligned} (W_\psi f)(b, a) &= \frac{\sqrt{a}}{2\pi} \left\{ \sum_{s=0}^{n-1} d_s \left( M[\widehat{\psi}(\omega); s+\lambda] + (-1)^{s+\lambda+1} \right. \right. \\ &\quad \left. \left. \times M[\widehat{\psi}(-\omega); s+\lambda] \right) a^{-s-\lambda} + \delta_n(a) \right\}, \end{aligned} \quad (30)$$

where

$$\begin{aligned} \delta_n(a) &= \lim_{\varepsilon \rightarrow 0+} \left( \int_0^\infty g_n(\omega) h(a\omega) e^{-\varepsilon\omega} d\omega \right. \\ &\quad \left. + \int_0^\infty g_n(-\omega) h(-a\omega) e^{-\varepsilon\omega} d\omega \right). \end{aligned} \quad (31)$$

Since  $g(\omega) = e^{ib\omega} \hat{f}(\omega)$ , the continuity of  $\hat{f}^{(m)}(\omega)$  implies continuity of  $g^{(m)}(\omega)$ . Using Theorem 1 we get the following existence theorem for formula (31).

**Theorem 2.** Assume that (i)  $\hat{f}^{(m)}(\omega)$  is continuous on  $(-\infty, \infty)$ , where  $m$  is a nonnegative integer; (ii)  $\hat{f}(\omega)$  has asymptotic expansion of the form (18) and the expansion is  $m$  times differentiable; (iii)  $\widehat{\psi}(\omega)$  satisfies (16) and (17) and (iv) as  $\omega \rightarrow \infty$ ,  $\omega^{-\beta} \hat{f}^{(j)}(\omega) = O(\omega^{-1-\varepsilon})$  for  $j = 0, 1, 2, \dots, m$  and for some  $\varepsilon > 0$ . Under these conditions, the result (30) holds with

$$\delta_n(a) = \frac{(-1)^m}{a^m} \int_{-\infty}^\infty g_n^{(m)}(\omega) (\overline{\widehat{\psi}(a\omega)})^{(-m)} d\omega, \quad (32)$$

where  $n$  is the smallest positive integer such that  $\lambda + n > m$ .

In the following sections we shall obtain asymptotic expansions for certain special cases of the general wavelet transform.

### 3 MORLET WAVELET TRANSFORM

In this section we choose

$$\psi(t) = e^{i\omega_0 t - t^2/2}.$$

Then from [1, p. 373] we have

$$\hat{\psi}(\omega) = \sqrt{2\pi} e^{\frac{-(\omega - \omega_0)^2}{2}},$$

which is exponentially decreasing. Therefore, Theorem1 is not directly applicable, but a slight modification of the technique works well. Assume that  $\hat{f}$  has an asymptotic expansion of the form (18). In this case we have

$$\begin{aligned} h(\omega) &= \overline{\hat{\psi}(\omega)} \\ &= \sqrt{2\pi} e^{\frac{-(\omega - \omega_0)^2}{2}} \end{aligned} \quad (33)$$

and

$$h(\omega) = O(1) \quad \text{as } \omega \rightarrow 0. \quad (34)$$

Then from (24) and (33), we get

$$\begin{aligned} I_1(a) &= \sum_{s=0}^{n-1} d_s M \left[ \sqrt{2\pi} e^{\frac{-(\omega - \omega_0)^2}{2}}; s + \lambda \right] a^{-s-\lambda} \\ &\quad + \lim_{\varepsilon \rightarrow 0+} \int_0^\infty g_n(\omega) \sqrt{2\pi} e^{\frac{-(a\omega - \omega_0)^2}{2}} e^{-\varepsilon\omega} d\omega, \end{aligned} \quad (35)$$

where

$$\begin{aligned} M \left[ \sqrt{2\pi} e^{\frac{-(\omega - \omega_0)^2}{2}}; s + \lambda \right] &= \sqrt{2\pi} \int_0^\infty \omega^{s+\lambda-1} e^{\frac{-(\omega - \omega_0)^2}{2}} d\omega \\ &= \sqrt{2\pi} e^{-\frac{\omega_0^2}{2}} \int_0^\infty \omega^{s+\lambda-1} e^{-\frac{\omega^2}{2} + \omega\omega_0} d\omega. \end{aligned}$$

Evaluating the last integral by means of formula [2, (31), p.320]:

$$\int_0^\infty x^{s-1} e^{-(x^2/2) - \beta x} dx = e^{(\beta^2/4)} \Gamma(s) D_{-s}(\beta), \quad \text{Re}(s) > 0,$$

where  $D_{-\nu}(x)$  denotes parabolic cylinder function, we get

$$M \left[ \sqrt{2\pi} e^{\frac{-(\omega - \omega_0)^2}{2}}; s + \lambda \right] = \sqrt{2\pi} e^{-\frac{\omega_0^2}{4}} \Gamma(s + \lambda) D_{-(s+\lambda)}(-\omega_0), \quad s + \lambda > 0. \quad (36)$$

From (35) and (36), we get

$$\begin{aligned} I_1(a) &= \sqrt{2\pi} e^{-\frac{\omega_0^2}{4}} \sum_{s=0}^{n-1} d_s \Gamma(s + \lambda) D_{-(s+\lambda)}(-\omega_0) a^{-s-\lambda} \\ &\quad + \int_0^\infty g_n(\omega) \sqrt{2\pi} e^{\frac{-(a\omega - \omega_0)^2}{2}} d\omega. \end{aligned} \quad (37)$$

Similarly, we get

$$I_2(a) = \sqrt{2\pi} e^{\frac{-\omega_0^2}{4}} \sum_{s=0}^{n-1} d_s \Gamma(s+\lambda) (-1)^{s+\lambda-1} D_{-(s+\lambda)}(\omega_0) a^{-s-\lambda} + \int_0^\infty g_n(-\omega) \sqrt{2\pi} e^{-\frac{(a\omega+\omega_0)^2}{2}} d\omega. \quad (38)$$

Finally, using (15), (37) and (38) we get

$$(W_\psi f)(b, a) = e^{\frac{-\omega_0^2}{4}} \sum_{s=0}^{n-1} d_s \Gamma(s+\lambda) [D_{-(s+\lambda)}(-\omega_0) + (-1)^{s+\lambda-1} D_{-(s+\lambda)}(\omega_0)] a^{-s-\lambda+\frac{1}{2}} + \delta_n(a), \quad (39)$$

where

$$\delta_n(a) = \sqrt{a} \int_0^\infty g_n(\omega) e^{-(a\omega-\omega_0)^2} d\omega + \sqrt{a} \int_0^\infty g_n(-\omega) e^{-(a\omega+\omega_0)^2} d\omega.$$

Using Theorem 2 we get the following existence theorem for formula (39).

**Theorem 3.** *Assume that  $\hat{f}(\omega)$  satisfies conditions of Theorem 2. Then the result (39) holds with*

$$\delta_n(a) = (-1)^m a^{-m+1/2} \int_{-\infty}^\infty g_n^{(m)}(\omega) \left( e^{\frac{-(a\omega-\omega_0)^2}{2}} \right)^{(-m)} d\omega,$$

where  $n$  is the smallest positive integer such that  $\lambda + n > m$ .

## 4 MEXICAN HAT WAVELET TRANSFORM

In this section we choose

$$\psi(t) = (1 - t^2) e^{-t^2/2}.$$

Then from [1, p.372]

$$h(\omega) := \hat{\psi}(\omega) = \sqrt{2\pi} \omega^2 e^{-\omega^2/2}; \quad (40)$$

so that

$$h(\omega) = O(\omega^2), \quad \omega \rightarrow 0. \quad (41)$$

Assume that  $\hat{f}$  has an asymptotic expansion of the form (18), and satisfies

$$\hat{f}(\omega) = O(e^{\sigma\omega^2}), \quad \omega \rightarrow +\infty, \quad (42)$$

for some  $\sigma > 0$ . Therefore,

$$g(\omega) := e^{ib\omega} \hat{f}(\omega) = O(e^{\sigma\omega^2}), \quad \omega \rightarrow +\infty. \quad (43)$$

Then by (23) and (40), we get

$$I_1(a) = \sum_{s=0}^{n-1} d_s M[\sqrt{2\pi}\omega^2 e^{-\omega^2/2}; s + \lambda] a^{-s-\lambda} + \delta_n^1(a), \quad (44)$$

where

$$\begin{aligned} M[\sqrt{2\pi}\omega^2 e^{-\omega^2/2}; s + \lambda] &= \sqrt{2\pi} \int_0^\infty \omega^{s+\lambda+1} e^{-\omega^2/2} d\omega \\ &= \sqrt{\pi} 2^{(s+\lambda+1)/2} \Gamma\left(\frac{s+\lambda+2}{2}\right), \end{aligned}$$

and

$$\delta_n^1 = \int_0^\infty g_n(\omega) \sqrt{2\pi} (a\omega)^2 e^{-(a\omega)^2/2} d\omega. \quad (45)$$

Similarly, we get

$$\begin{aligned} I_2(a) &= \sqrt{\pi} 2^{(\lambda+1)/2} \sum_{s=0}^{n-1} d_s (-1)^{s+\lambda-1} 2^{s/2} \Gamma\left(\frac{s+\lambda+2}{2}\right) a^{-s-\lambda} \\ &\quad + \delta_n^2(a), \end{aligned} \quad (46)$$

where

$$\delta_n^2(a) = \int_0^\infty g_n(-\omega) \sqrt{2\pi} (a\omega)^2 e^{-(a\omega)^2/2} d\omega. \quad (47)$$

Finally, using (15), (44) and (46), we have

$$\begin{aligned} (W_\psi f)(b, a) &= \frac{2^{(\lambda+1)/2}}{\sqrt{\pi}} \sum_{s=0}^{n-1} d_s 2^{s/2} \Gamma\left(\frac{s+\lambda+2}{2}\right) \{1 + (-1)^{s+\lambda-1}\} \\ &\quad \times a^{-s-\lambda+1/2} + \delta_n(a), \end{aligned} \quad (48)$$

where

$$\delta_n(a) = 2^{3/2} \sqrt{\pi} \int_0^\infty g_n(\omega) (a\omega)^2 e^{-(a\omega)^2/2} d\omega.$$

Existence theorem for formula (48) is as follows:

**Theorem 4.** Assume that  $\hat{f}(\omega)$  satisfies conditions of Theorem 2. Then the result (48) holds with

$$\delta_n(a) = \left(\frac{-1}{a}\right)^m 2^{3/2} \sqrt{\pi} \int_{-\infty}^\infty g_n^{(m)}(\omega) ((a\omega)^2 e^{-(a\omega)^2/2})^{(-m)} d\omega,$$

where  $n$  is the smallest positive integer such that  $\lambda + n > m$ .



## 5 Haar wavelet transform

In this section we choose

$$\psi(t) = \begin{cases} 1, & 0 \leq t < 1/2 \\ -1, & 1/2 \leq t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then from [1, p.368],

$$\widehat{\psi}(\omega) = 4ie^{-i\omega/2} \frac{\sin^2 \omega/4}{\omega} = \frac{i}{\omega} (1 - 2e^{i\omega/2} + e^{i\omega}). \quad (49)$$

Although the condition  $\beta > 0$  of (16) is not satisfied in this case but the result (6)- (7) remains valid, cf. [3, p.753].

Clearly,

$$h(\omega) = O(\omega), \quad \text{as } \omega \rightarrow 0. \quad (50)$$

Assume that  $\hat{f}(\omega)$  has an asymptotic expansion of the form (18). Using (15) and (49) we get

$$\begin{aligned} I_1(a) &= \int_0^\infty e^{ib\omega} \hat{f}(\omega) \frac{1}{a\omega} (1 - 2e^{ia\omega/2} + e^{ia\omega}) d\omega \\ &= \frac{i}{a} F(b) - 2i \int_0^\infty e^{ib\omega} \hat{f}(\omega) \frac{e^{ia\omega/2}}{a\omega} d\omega \\ &\quad + i \int_0^\infty e^{ib\omega} \hat{f}(\omega) \frac{e^{ia\omega}}{a\omega} d\omega, \end{aligned}$$

where

$$F(b) = \int_0^\infty e^{ib\omega} \frac{\hat{f}(\omega)}{\omega} d\omega.$$

Then, (20) and the generalized Mellin transform formula [4, Lemma 2, p.198]:

$$M[e^{it}; z] = e^{i\pi z/2} \Gamma(z)$$

we get

$$\begin{aligned} I_1(a) &= \frac{i}{a} F(b) - \frac{2i}{a} \int_0^\infty \left[ \sum_{s=1}^\infty d_s \omega^{s+\lambda-2} + g_n(\omega) \right] e^{ia\omega/2} d\omega \\ &\quad + \frac{i}{a} \int_0^\infty \left[ \sum_{s=1}^\infty d_s \omega^{s+\lambda-2} + g_n(\omega) \right] e^{ia\omega} d\omega \\ &= \frac{i}{a} F(b) + \frac{2i}{a} \left\{ \sum_{s=1}^{n-1} d_s \Gamma(s + \lambda - 1) (a/2)^{-s-\lambda+1} e^{i\pi(s+\lambda)/2} \right. \\ &\quad \left. - (2i/a)^n \int_0^\infty g_n^{(n)}(\omega) e^{ia\omega/2} d\omega \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{i}{a} \left\{ - \sum_{s=1}^{n-1} d_s \Gamma(s + \lambda - 1) (a)^{-s-\lambda+1} e^{i\pi(s+\lambda)/2} \right. \\
& \left. + (i/a)^n \int_0^\infty g_n^{(n)}(\omega) e^{ia\omega} d\omega \right\} \\
& = \frac{i}{a} F(b) + i \sum_{s=1}^{n-1} d_s \Gamma(s + \lambda - 1) a^{-s-\lambda} (2^{s+\lambda} - 1) e^{i\pi(s+\lambda)/2} \\
& \quad + (i/a)^{n+1} \int_0^\infty g_n^{(n)}(\omega) (e^{ia\omega} - 2^{n+1} e^{ia\omega/2}) d\omega.
\end{aligned} \tag{51}$$

Notice that for existence of the Mellin transform in the above case we have to assume that  $d_0 = 0$ . Similarly,

$$\begin{aligned}
I_2(a) &= \frac{i}{a} \int_{-\infty}^0 e^{ib\omega} \frac{\hat{f}(\omega)}{\omega} d\omega + i \sum_{s=1}^{n-1} d_s \Gamma(s + \lambda - 1) a^{-s-\lambda} (-1)^{s+\lambda-1} \\
&\quad \times (2^{s+\lambda} - 1) e^{i\pi(s+\lambda)/2} + (i/a)^{n+1} \int_0^\infty g_n^{(n)}(-\omega) \\
&\quad \times (e^{-ia\omega} - 2^{n+1} e^{-ia\omega/2}) d\omega.
\end{aligned} \tag{52}$$

Finally, using formula [2, (15), p.152], from (15), (51) and (52) we get

$$\begin{aligned}
(W_\psi f)(b, a) &= \frac{i}{\sqrt{a}} f^{(-1)}(b) + \frac{i}{\pi} \sum_{s=0}^{n-1} d_s \Gamma(s + \lambda - 1) a^{-s-\lambda+1/2} \\
&\quad \times \{1 + (-1)^{s+\lambda-1}\} (2^{s+\lambda} - 1) e^{i\pi(s+\lambda)/2} + \delta_n(a),
\end{aligned} \tag{53}$$

where

$$f^{(-1)}(b) = (D^{-1}f)(b)$$

and

$$\delta_n(a) = (i/a)^{n+1} \frac{\sqrt{a}}{2\pi} \int_0^\infty g_n^{(n)}(\omega) (e^{ia\omega} - 2^{n+1} e^{ia\omega/2}) d\omega. \tag{54}$$

Existence theorem for (53) is as follows:

**Theorem 5.** Assume that  $\hat{f}(\omega)$  satisfies conditions of Theorem 2. Then the result (53) holds with

$$\delta_n(a) = (i/a)^{m+1} \frac{\sqrt{a}}{2\pi} \int_{-\infty}^\infty g_n^{(m)}(\omega) (e^{ia\omega} - 2^{m+1} e^{ia\omega/2}) d\omega,$$

where  $n$  is the smallest positive integer such that  $\lambda + n > m$ .

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